

PERIODIC BILLIARD ORBITS ON n -DIMENSIONAL ELLIPSOIDS WITH IMPACTS ON CONFOCAL QUADRICS AND ISOPERIODIC DEFORMATIONS.

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ABSTRACT. In our paper we study periodic geodesic motion on multidimensional ellipsoids with elastic impacts along confocal quadrics. We show that the method of isoperiodic deformation is applicable.

1. INTRODUCTION

One of the classical integrable problems is the geodesic motion on the ellipsoid. Complete integrability of the 3-dimensional problem was proven by Jacobi [25]. Explicit formulas in terms of genus 2 θ -functions were found by Weierstrass [44]. Integrability of the multidimensional problem was established by Moser [32]. Connections with the Neumann problem were exploited by Knörrer [27]. Integrability of the Neumann problem has a very transparent explanation, based on the following observation by Trubowitz and Moser (see [32]): taking properly normalized eigenfunctions at the right ends of gaps for the $(N - 1)$ -gap 1-dimensional stationary Schrödinger operator we obtain the coordinates of the particles.

The isomorphism between the geodesics on the ellipsoid and the Neumann problem is nontrivial, because it includes a nonlinear change of independent variable. The motion on the Jacobian for the Neumann problem is uniform, and the motion for the geodesics on the ellipsoid is along straight lines, but the speed in the natural parameter is not constant. It means, that the analog of the Trubowitz-Moser isomorphism maps this dynamics into the Harry-Dym equation instead of the Korteweg-de Vries (KdV) equation. On the level of the associated

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spectral problems it means, that one has to consider the 1-dimensional acoustic problem

$$(1) \quad -\partial_t^2 \Psi(E, t) = Eu(t)\Psi(E, t), \quad E = 1/\lambda$$

instead of the 1-dimensional Schrödinger problem

$$(2) \quad -\partial_\sigma^2 \Psi(E, \sigma) + \tilde{u}(\sigma)\Psi(E, \sigma) = E\Psi(E, \sigma).$$

The connection between the geodesics on the ellipsoid and the Harry-Dym equation was first observed by Cao [8], and independently by Veselov [41], [42].

From the geodesic motion on the ellipsoid one can construct an integrable billiard motion taking the geodesic dynamics on the n -dimensional ellipsoid

$$(3) \quad Q = \left\{ \frac{x_1^2}{a_1} + \dots + \frac{x_{n+1}^2}{a_{n+1}} = 1 \right\},$$

combined with elastic reflections along one or more confocal quadrics

$$Q_{d_s} = \left\{ \frac{x_1^2}{a_1 - d_s} + \dots + \frac{x_{n+1}^2}{a_{n+1} - d_s} = 1 \right\}, \quad s = 1, \dots, r.$$

d_s being arbitrary fixed parameters. An amazing property of this motion is that all conservation laws remain invariant after reflections, therefore this dynamical system is also completely integrable. Such property was first observed by Chang and Shi [9] for the triaxial ellipsoids, while its multidimensional generalization was obtained in [10]. As a consequence the trajectories between impacts (arches of geodesic) or their continuations are tangent to the same set of $n - 1$ quadrics (caustics) in \mathbb{R}^{n+1} that are confocal to Q and Q_{d_s} . In the same papers it was proven the Poncelet property: if one trajectory is closed after p bounces, all the trajectories sharing the same constants of motion are also closed after p bounces and have the same length.

The billiard on the ellipsoid is described by the maps $\mathcal{B}_s : (x^{(N)}, v_{\text{out}}^{(N)}) \mapsto (x^{(N+1)}, v_{\text{out}}^{(N+1)})$, where $x^{(N)}, v_{\text{out}}^{(N)} \in \mathbb{R}^{n+1}$, $N \in \mathbb{N}$, are respectively the coordinates of the N -th impact point on $Q \cap Q_{d_s}$ and the outgoing velocity at this point, whereas $x^{(N+1)}, v_{\text{out}}^{(N+1)}$ denote the same objects at the next impact point. This billiard map is transcendental in the following sense: the shift takes place on a non-Abelian variety and, as a result, it is not fixed. This property was first observed by Veselov [40] for the billiard on triaxial ellipsoids and generalized for higher dimensions by Abenda and Fedorov in [3]. More precisely, the generic complex invariant manifolds are not Abelian varieties, but open subsets of theta-divisors of n -dimensional hyperelliptic Jacobians of a

given hyperelliptic curve Γ . Thus, in the real domain, the new values $(x^{(N+1)}, v_{\text{out}}^{(N+1)})$ are determined by solving a system of transcendental equations which involve the inversion of hyperelliptic integrals, whereas in the complex coordinates the billiard map is infinitely many valued.

We stress that all of these properties stand in sharp contrast to both the Birkhoff ($p = 0$) [7, 33, 38, 39] and the Hooke ($p = 2$) [18] billiard systems, where the billiard map is given by a shift by a constant vector on a complex invariant manifold which is an open subset of an Abelian variety. Due to the above properties, the billiard maps \mathcal{B}_s can be regarded as a discrete analog of the algebraic integrable systems with deficiency studied, in particular, in [43, 2, 4, 16]). The latter systems, although being Liouville integrable, are not algebraic completely integrable: their generic complex solutions have movable algebraic branch points and are single-valued on an infinitely sheeted ramified covering of the complex plane t , so they possess the so called weak Painlevé property. The branching of the billiard maps \mathcal{B}_s can be viewed as a discrete version of such a property (see also [21]).

The development of the finite-gap approach to the 1-d Schrödinger operator (2) and the KdV equation was started by Novikov [34]. Generic finite-gap potentials are quasiperiodic. Selection of purely periodic potentials is rather nontrivial.

The principal periodicity constraints discussed in the literature are the following:

- (1) Periodicity in one real variable.
- (2) Periodicity in two real variables.
- (3) Double periodicity in one complex variable. Such solutions are called elliptic solitons.

The problem of selecting space-periodic potentials for the 1-D Schrödinger operator was solved by Marchenko and Ostrovski [31] in terms of conformal maps. It is also possible to study the variety of all spectral curves generating purely periodic potentials, using the so-called isoperiodic deformations (they can be interpreted as Loewner equations for these conformal maps). Period preserving deformations first arose in the papers by Ercolani, Forest, McLaughlin and Sinha [17], Krichever [30], Schmidt [35]. Grinevich and Schmidt [22] suggested to use isoperiodic deformations for selecting pure periodic solutions, they also showed that these flows can be transformed to a very simple algebraic form by extending the phase space.

The finite-gap theory for the acoustic problem (1) was developed by Dmitrieva and collaborators [11], [12], [13]. This problem can be transformed to the Schrödinger problem (2) by a reciprocal transformation

over the independent variable. Equations for the isoperiodic deformations of this problem were written by D.Zakharov [45]. These results can be automatically applied to the periodic geodesics.

The first non-trivial examples of elliptic solitons for the KdV equation (and 1-D Schrödinger operators potentials respectively) were found by Dubrovin and Novikov [15]. These solutions correspond to genus $g = 2$ curves. Airault, McKean and Moser [6] established the connection of KdV elliptic solitons with a special reduction of the elliptic Calogero-Moser system valid for arbitrary genera. An analogous connection between the Kadomtsev-Petviashvili (KP) equation and the full elliptic Calogero-Moser system was discovered by Krichever [29]. A complete characterization of the double-periodic in the complex x -variable KdV and KP solutions in terms of so-called tangential coverings was obtained by Treibich and Verdier [36], [37]. Due to the Trubowitz-Moser isomorphism these results can be also applied to fully characterize and construct real geodesics and real billiard trajectories, which are double-periodic in the complex length [19], [3], [1].

It is natural to pose the problem of constructing real billiard trajectories on n -dimensional ellipsoids which are simply-periodic. In our paper we show, that the isoperiodic technique can be applied to this problem.

2. GEODESICS ON ELLIPSOIDS

In this section, to settle notations, we briefly outline the description of the geodesics and billiard motions on a quadric. The geodesic motion on an n -dimensional ellipsoid Q is well known to be integrable and to be linearized on a covering of the Jacobian of a genus n hyperelliptic curve [32]. Namely, let t be the natural parameter of the geodesic and $\lambda_1, \dots, \lambda_n$ be the ellipsoidal coordinates on Q defined by the formulas

$$(4) \quad x_i = \sqrt{\frac{a_i(a_i - \lambda_1) \cdots (a_i - \lambda_n)}{\prod_{j \neq i} (a_i - a_j)}}, \quad i = 1, \dots, n+1.$$

Then, denoting $\dot{\lambda}_k = d\lambda_k/dt$ the corresponding velocities, the total energy $\frac{1}{2}(\dot{x}, \dot{x})$ takes the Stäckel form

$$H = -\frac{1}{8} \sum_{k=1}^n \left[\frac{\prod_{j \neq k}^n (\lambda_k - \lambda_j)}{\prod_{j=1}^{n+1} (\lambda_k - a_j)} \right] \lambda_k \dot{\lambda}_k^2.$$

According to the Stäckel theorem, the system is Liouville integrable. After fixing the constant of motion c_k , $k = 1, \dots, n-1$ and assuming $H = 1/2$ we obtain Dubrovin equations:

$$(5) \quad \dot{\lambda}_k = -2 \frac{w_k}{\lambda_k \prod_{j \neq k}^n (\lambda_k - \lambda_j)}.$$

Here

$$(6) \quad w_k^2 = -\lambda_k(\lambda_k - a_1) \cdots (\lambda_k - a_{n+1})(\lambda_k - c_1) \cdots (\lambda_k - c_{n-1}), \quad k = 1, \dots, n.$$

The evolution of the divisor λ_k is defined on the genus n hyperelliptic curve

$$(7) \quad \Gamma : \left\{ w^2 = -\lambda \prod_{i=1}^{n+1} (\lambda - a_i) \prod_{k=1}^{n-1} (\lambda - c_k) \right\} = \left\{ w^2 = -\prod_{i=0}^{2n} (\lambda - b_i) \right\},$$

where we set the following notation throughout the paper

$$(8) \quad \{0, a_1 < \cdots < a_{n+1}, c_1 < \cdots < c_{n-1}\} = \{b_0 = 0 < b_1 < \cdots < b_{2n}\}.$$

The constants of motion c_k have the following geometrical meaning (see [32]): the corresponding geodesics are tangent to the quadrics $Q_{c_1}, \dots, Q_{c_{n-1}}$ of the confocal family $Q_c = \{x_1^2/(a_1 - c) + \cdots + x_{n+1}^2/(a_{n+1} - c) = 1\}$.

In particular, the reality condition for geodesics on ellipsoids is equivalent to either $c_i = b_{2i}$ or $c_i = b_{2i+1}$, $i = 1, \dots, n-1$ (see [26, 5]). Since we are interested in the reality problem, we take, without loss of generality, $b_{2i-1} < \lambda_i < b_{2i}$, $i = 1, \dots, n$.

Let us recall (see [8], [41]) how to reconstruct the geodesic motion from the solutions of the acoustic problem (1). The n -gap eigenfunctions of (1) are defined by the following properties:

- (1) For any t , $\Psi(\gamma, t)$ is meromorphic in γ on the spectral curve $\Gamma \setminus \{\lambda = 0\}$.
- (2) For any t , $\Psi(\gamma, t)$ has n simple poles at the points $\gamma_1(0), \dots, \gamma_n(0)$. Here $\lambda_j(t)$ is the projection of $\gamma_j(t)$ to the λ -plane.
- (3) There exists a function $\sigma(t)$ such, that

$$\exp\left(-\frac{i\sigma(t)}{\sqrt{\lambda}}\right) \Psi(\gamma, t)$$

is regular near the point $\lambda = 0$.

(4)

$$(9) \quad \Psi(\lambda, t) = 1 + \frac{it}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \rightarrow \infty.$$

Let us define

$$(10) \quad x_j(t) = \Psi(a_j, t) \sqrt{\frac{a_j \prod_{k=1}^n (a_j - \lambda_k(0))}{\prod_{k \neq j} (a_j - a_k)}}$$

Consider the following pair of meromorphic forms in the λ -plane:

$$(11) \quad \Omega_1 = \Psi(\gamma, t) \Psi(\tau\gamma, t) \frac{\prod_{k=1}^n (\lambda - \lambda_k(0))}{\prod_{k=1}^{n+1} (\lambda - a_k)} d\lambda,$$

$$(12) \quad \Omega_2 = \dot{\Psi}(\gamma, t) \dot{\Psi}(\tau\gamma, t) \frac{\lambda \prod_{k=1}^n (\lambda - \lambda_k(0))}{\prod_{k=1}^{n+1} (\lambda - a_k)} d\lambda.$$

Here τ denotes the hyperelliptic involution. From (9) it follows, that the integrals of Ω_1, Ω_2 over sufficiently large contour are equal to $2\pi i$. The residues of Ω_1 and Ω_2 at the point a_j are $\frac{x_j^2(t)}{a_j}$ and $\dot{x}_j^2(t)$ respectively. Therefore we obtain (3) together with the normalization condition:

$$(13) \quad \sum_{j=1}^{n+1} \dot{x}_j^2(t) = 1.$$

The motion is governed by the Lagrange multipliers principle:

$$\ddot{x}_j(t) = -u(t) \frac{x_j(t)}{a_j}$$

After the re-parameterization

$$(14) \quad dt = \frac{\lambda_1 \cdots \lambda_n}{\sqrt{b_1 b_2 \cdots b_{2n}}} d\sigma,$$

(this re-parameterization coincides with the change of independent variables, connecting the acoustic operator (1) with Schrödinger operator (2)) the evolution of λ_i is described by quadratures

$$(15) \quad \sqrt{\prod_{j=1}^{2g} b_j} \cdot \left[\frac{\lambda_1^{k-1}}{2w_1} d\lambda_1 + \cdots + \frac{\lambda_n^{k-1}}{2w_n} d\lambda_n \right] = \begin{cases} d\sigma & \text{for } k = 1, \\ 0 & \text{for } k = 2, \dots, n, \end{cases}$$

The quadratures involve n independent holomorphic differentials on Γ .

Let α_i, β_i , $i = 1, \dots, n$ be the conventional homological basis depicted in Figure 1 and

$$(16) \quad \omega_j = \sqrt{\prod_{j=1}^{2g} b_j} \cdot \frac{\lambda^{j-1} d\lambda}{w}, \quad j = 1, \dots, n,$$

be the usual basis of holomorphic differentials. (15) give rise to the Abel–Jacobi map of the n -th symmetric product $\Gamma^{(n)}$ to the Jacobian variety of Γ (see for example [23]):

$$(17) \quad u_j = \int_{P_0}^{P_1} \omega_j + \dots + \int_{P_0}^{P_n} \omega_j = \begin{cases} 2\sigma + \text{const.}, & \text{for } j = 1, \\ \text{const.}, & \text{for } j = 2, \dots, n, \end{cases}$$

where and P_0 is a fixed basepoint and $P_k = (\lambda_k, w_k) \in \Gamma$, $k = 1, \dots, n$. Then, the geodesic motion in the new parameterization is linearized on the Jacobian variety of Γ . Its complete theta-functional solution was presented in [44] ($n = 2$), and in [26] ($n > 2$), whereas a topological classification of real geodesics on quadrics was made in [5] and an algebraic geometric characterization of a dense set of real closed geodesics on ellipsoids has been given in [1].

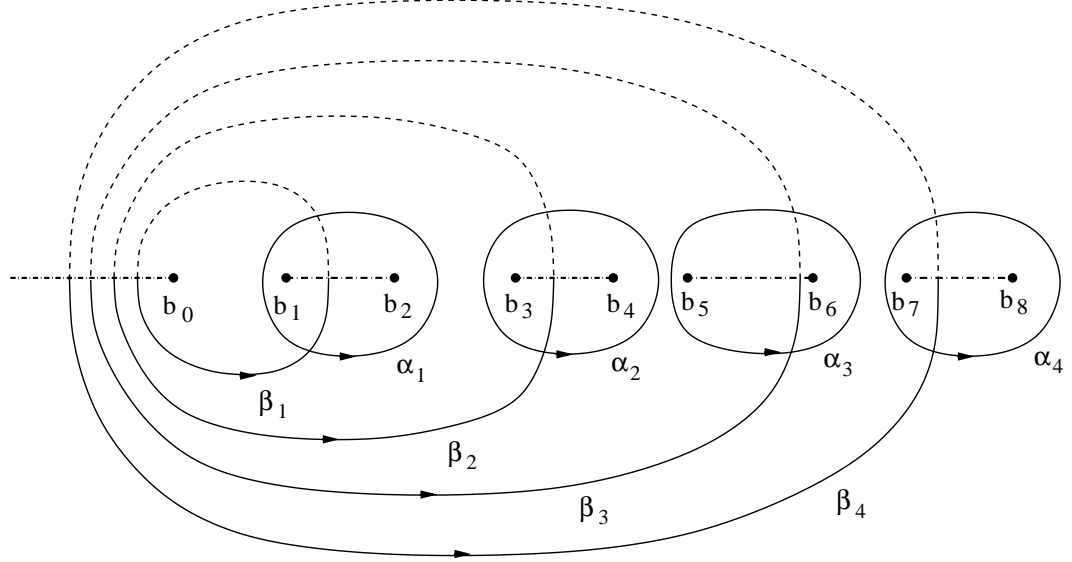


Figure 1: The canonical basis of cycles.

3. THE BILLIARDS ON THE ELLIPSOID Q .

The billiard map is associated to the geodesic motion of a point on Q with elastic bounces along a set of confocal quadrics Q_{d_s} , $s = 1, \dots, r$,

$d_1 < d_2 < \dots < d_r$. We are interested in real motions only, and we assume, that all reflectors $Q \cap Q_{d_s}$ are met at least once. Therefore we have the following constraints to the positions of the reflectors:

- (1) The points d_s are located inside gaps: $d_s \in \bigcup_{j=1}^n]b_{2j-1}, b_{2j}[$, $s = 1, \dots, r$
- (2) Each gap $[b_{2j-1}, b_{2j}]$ contains at most 2 points d_s .
- (3) If the gap $[b_{2j-1}, b_{2j}]$ contains exactly 2 points d_s, d_{s+1} , then the projection of the divisor point $\lambda_j(t)$ is located inside the interval $[d_s, d_{s+1}]$.
- (4) If the gap $[b_{2j-1}, b_{2j}]$ contains exactly 1 point d_s , then the projection of the divisor point $\lambda_j(t)$ remains forever inside one of the following intervals: $[b_{2j-1}, d_s]$ or $[d_s, b_{2j}]$.

The divisor dynamics can be described in the following way. It is guided by the Dubrovin equations (5) between the subsequent collisions with the reflectors. If a divisor point λ_j collides with reflector d_s , it “jumps” to the opposite sheet of Γ with respect to the hyperelliptic involution:

$$(18) \quad \tau : (\lambda_j, w_j) \rightarrow (\lambda_j, -w_j).$$

Each point d_s has two preimages in the spectral curve Γ : $D_s^+ = (d_s, +w(d_s))$, $D_s^- = (d_s, -w(d_s))$, $w(d_s) > 0$. Let us introduce the following notation:

$$(19) \quad D_s^{\text{in}} = \begin{cases} D_s^+ & \text{if } \frac{(\lambda_j - d_s)}{\prod_{k \neq j} (\lambda_j - \lambda_k)} > 0 \\ D_s^- & \text{if } \frac{(\lambda_j - d_s)}{\prod_{k \neq j} (\lambda_j - \lambda_k)} < 0 \end{cases}$$

$$(20) \quad D_s^{\text{out}} = \tau D_s^{\text{in}}$$

From the Dubrovin equations (5) it follows, that the divisor points always jump from D_s^{in} to D_s^{out} .

Under the Abel–Jacobi map (17), the condition $\lambda_j = d_s$ defines two codimension one subvarieties Θ_s^{in} and Θ_s^{out} in $\text{Jac}(\Gamma)$, and each reflection on the $Q \cap Q_{d_s}$ corresponds to the jump from one of these subvarieties to the other one.

Hence, for fixed constants of motion, the coordinates x of impact points on $Q \cap Q_{d_s}$ with velocities v are described by degree $n - 1$ divisors $\{P_1, \dots, P_{n-1}\}$, that is, by a point φ on Θ_s^{in} or Θ_s^{out} .

Let us denote

$$(21) \quad \mathbf{q}_s = (\mathbf{q}_s^1, \dots, \mathbf{q}_s^n)^T = \int_{D_s^{\text{in}}}^{D_s^{\text{out}}} (\omega_1, \dots, \omega_n)^T \in \mathbb{R}^n.$$

Let us recall the situation in case of one reflector $d_1 = d$, $\Theta_1^{\text{in}} = \Theta^{\text{in}}$, $\Theta_1^{\text{out}} = \Theta^{\text{out}}$, $D^{\text{in}} = D_1^{\text{in}}$, $D^{\text{out}} = D_1^{\text{out}}$, $\mathbf{q} = \mathbf{q}_1$. The billiard map is transcendental, as it was first observed by Veselov for $n = 2$ [40]. An algebraic geometric description of the motion between impacts and at the elastic bounce along $Q \cap Q_d$ is given by the following proposition [3].

Theorem 3.1. *The geodesic motion on Q between subsequent impact points with coordinate $x^{(N)}$ and $x^{(N+1)}$ on $Q \cap Q_d$ corresponds to the straight line uniform motion on $\text{Jac}(\Gamma)$ between Θ^{out} and Θ^{in} along the u_1 -direction. The point of intersection with Θ^- gives the next coordinate $x^{(N+1)}$ and the ingoing velocity $v_{\text{in}}^{(N+1)}$. At the point $x^{(N+1)}$ the reflection $v_{\text{in}}^{(N+1)} \mapsto v_{\text{out}}^{(N+1)}$ (from the ingoing to the outgoing velocity) results in jumping back from Θ^{in} to Θ^{out} by the shift vector \mathbf{q} , which does not change $x^{(N+1)}$. Then the procedure iterates.*

The process is sketched in Figure 2. Here A_{in}^N denotes the image through the Abel map of the divisor, associated to the point $(x^{(N)}, v_{\text{in}}^{(N)})$, A_{out}^N denotes the image of $(x^{(N)}, v_{\text{out}}^{(N)})$ respectively:

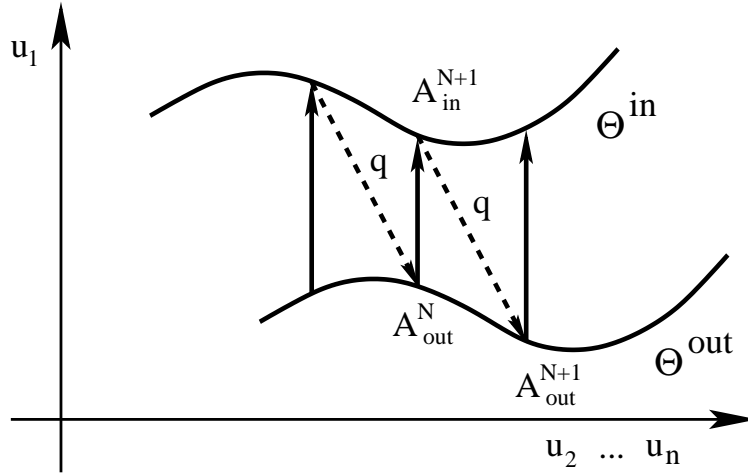


Figure 2: The billiard dynamics in the Jacobian.

To study the billiard dynamics it is important to choose an appropriate canonical basis of cycles. We assume that Γ has the following system of cuts: an infinite cut $]-\infty, 0]$ and finite cuts over gaps $[b_{2j-1}, b_{2j}]$ (all cuts are real). We choose the α -cycles in the standard way (see Figure 3). It is important that they do not intersect the cuts, and

consequently, the trajectories of the divisor. If the gap $[b_{2j-1}, b_{2j}]$ does not contain reflectors, the corresponding β_j -cycle is also chosen in the standard way, and it intersects exactly once the cut $[b_{2j-1}, b_{2j}]$. If the gap $[b_{2j-1}, b_{2j}]$ contains at least one reflector, we choose the cycle β_j so, that it does not intersect the divisor trajectory. In the case of two reflectors in the gap $[b_{2j-1}, b_{2j}]$ there are two possible choices of β_j (see cycle β_3 in Figure 3).

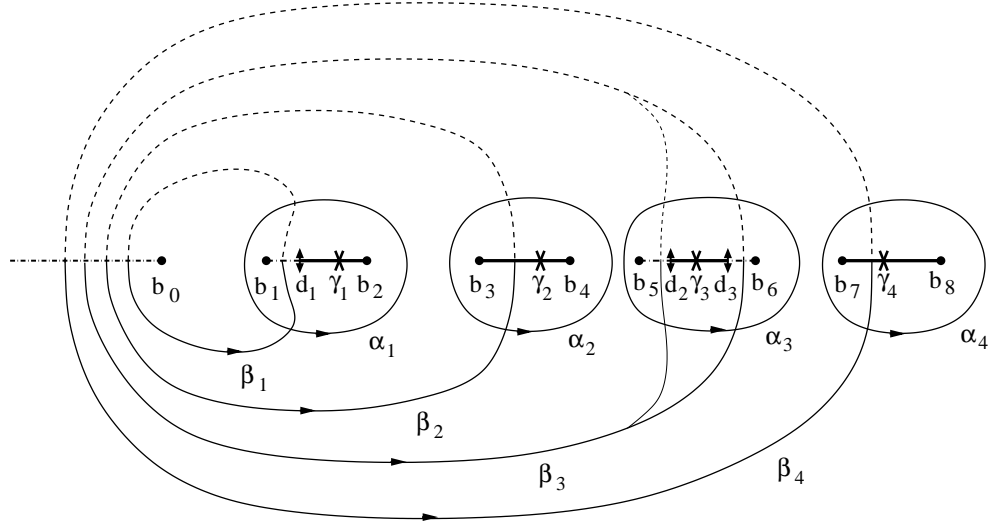


Figure 3: The canonical basis of cycles in the presence of reflectors.

Let us also introduce an auxiliary set of curves $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n$. If the gap $[b_{2j-1}, b_{2j}]$ does not contain reflectors, $\tilde{\alpha}_j$ is the preimage of $[b_{2j-1}, b_{2j}]$ with the proper orientation and homologically $\tilde{\alpha}_j \sim \alpha_j$. If the gap $[b_{2j-1}, b_{2j}]$ contains reflectors, $\tilde{\alpha}_j$ is the part of the preimage of $[b_{2j-1}, b_{2j}]$ where the divisor moves. The curves $\tilde{\alpha}_j$ inherit the orientations from the basic cycles α_j .

Let us define the auxiliary basis $\tilde{\omega}_1, \dots, \tilde{\omega}_n$ of holomorphic differentials with the following **non-standard** normalization:

$$(22) \quad \int_{\tilde{\alpha}_j} \tilde{\omega}_k = \delta_{jk}.$$

Proposition 3.2. *A basis $\tilde{\omega}_1, \dots, \tilde{\omega}_n$ with normalization (22) exists and is unique.*

The existence and the uniqueness of such basis of holomorphic differentials, follows immediately from the following lemma.

Lemma 3.3. *Let $\mathcal{G}_{(1)}, \dots, \mathcal{G}_{(N)}$, $\mathcal{G} = \cup \mathcal{G}_{(j)}$ be a set of paths in the cycles α_k such, that*

- (1) The velocity vector for all paths $\mathcal{G}_{(j)}$ is everywhere non-zero.
- (2) All paths $\mathcal{G}_{(j)}$ belonging to the same α_k have the same orientation with respect to the standard orientation on α_k . Let us denote: $\mathcal{G}_k = \mathcal{G} \cap \alpha_k$, i.e. \mathcal{G}_k is the union of all trajectories, lying in the cycle α_k .

Then for any k such, that $\mathcal{G}_k \neq \emptyset$ there exists a holomorphic differential ω such, that

$$(23) \quad \int_{\mathcal{G}_i} \omega = \delta_{kl}.$$

Proof of Lemma 3.3: Let us consider the following holomorphic differential

$$(24) \quad \omega = \mathcal{C} \frac{\prod_{j \neq k} (\lambda - \tilde{\eta}_j) d\lambda}{w},$$

where $\tilde{\eta}_j \in \alpha_j$. Differential ω has a constant sign on α_k , therefore

$$(25) \quad \int_{\mathcal{G}_k} \omega \neq 0.$$

If $\mathcal{G}_j = \emptyset$ then

$$(26) \quad I_j = \int_{\mathcal{G}_j} \omega = 0.$$

If $\mathcal{G}_j \neq \emptyset$ and $\tilde{\eta}_j$ lies in the end of the interval $[b_{2j-1}, b_{2j}]$ then integral I_j is nonzero. The signs of I_j for $\tilde{\eta}_j = b_{2j-1}$ and $\tilde{\eta}_j = b_{2j}$, are opposite, and they do not depend on the position of other zeroes $\tilde{\eta}_i$. Therefore we can tune the set $\tilde{\eta}_j$ so, that all $I_j = 0$ for all $j \neq k$. To complete the proof is it sufficient to choose a proper normalization constant \mathcal{C} .

Remark 3.4. *This proof uses the following topological Lemma: Let $f = (f^1(x_1, \dots, x_n), \dots, f^n(x_1, \dots, x_n))$ be a map from the n -dimensional cube I^n : $-1 \leq x_k \leq 1$, $k = 1, \dots, n$ to the n -dimensional space \mathbb{R}^n with the following property: for all k $f^k(\vec{x}) < 0$ at the face $x_k = -1$ and $f^k(\vec{x}) > 0$ at the face $x_k = 1$. Then the preimage of the origin $\vec{x} = (0, \dots, 0)$ is non-empty.*

The proof of the Lemma is based on standard topological arguments and we do not present it here.

Corollary 3.5. *Let $\hat{\omega}_1, \dots, \hat{\omega}_n$ be the **standard** basis of holomorphic differentials*

$$(27) \quad \int_{\alpha_j} \hat{\omega}_k = \delta_{jk},$$

\mathcal{J} be a subset of the set $\{1, 2, \dots, n\}$ such, that for any $k \in \mathcal{J}$ $\mathcal{G}_k \neq \emptyset$. Then the matrix

$$(28) \quad \mathcal{A}_{kl} = \int_{\mathcal{G}_k} \hat{\omega}_l, \quad k, l \in \mathcal{J}$$

is non-degenerate.

To prove this Corollary it is sufficient to assume, that for all k such, that $1 \leq k \leq n$, $k \notin \mathcal{J}$ $\mathcal{G}_k = \alpha_k$ and to apply Lemma 3.3. The differential obtained has zero integral over all contours α_k , $k \notin \mathcal{J}$, therefore it is a linear combination of $\hat{\omega}_l$, $l \in \mathcal{J}$.

Periodic billiard trajectories. First of all, let us recall, that there are two different descriptions of periodicity for the billiard motion. If the billiard trajectory is closed, it implies, that the divisor motion is periodic with the same period. From the periodicity of the divisor motion it follows, that the motion on the ellipsoid is also periodic, but the period may be greater, since the same divisor corresponds to 2^{n+1} different points in the phase space assuming all constants of motion to be fixed. These points can be obtained one from another by the reflections $x_j \rightarrow -x_j$. We see, that each periodic divisor motion corresponds to 2^{n+1} different periodic trajectories.

In the following we study necessary and sufficient conditions for the periodicity of the divisor motion.

Let us recall the necessary and sufficient condition on the spectral data to generate periodic billiard trajectories. In our setting for arbitrary number of reflectors we have:

Lemma 3.6. *The billiard trajectory is periodic if and only if there exist a collection of integers m_1, \dots, m_n such that*

$$(29) \quad \sum_{j=1}^n m_j \int_{\tilde{\alpha}_j} \omega_k = 0, \quad k = 2, \dots, n.$$

In such a case the length of the billiard trajectory in the rescaled variable σ is given by

$$(30) \quad \sum_{j=1}^n m_j \int_{\tilde{\alpha}_j} \omega_1 = 2L.$$

Here ω_k is the basis of holomorphic differentials (16) normalized by the asymptotics near the Weierstrass point $\lambda = 0$.

The above conditions were also given in [14] in an equivalent formulation and are in general transcendental in the parameters of the problem (the square semiaxes, the constants of the motion and the position of reflectors). In the case of one reflector, in [3] the authors gave a sufficient condition so that the periodicity condition for the billiard trajectory is algebraic in the parameters.

To apply the method of isoperiodic deformations we need an alternative characterization of the periodic spectral data. The remaining part of this Section is dedicated to this alternative characterization in terms of the so-called generalized quasimomentum $d\tilde{p}$.

First of all we assume that our billiard motion is periodic, and we construct the generalized quasimomentum, satisfying the properties 1-9 in Lemma 3.10. Then we show that a unique generalized quasimomentum satisfying properties 1-7 in Lemma 3.10 exists for general quasiperiodic solutions. Finally we check, that the motion is periodic if and only if the properties 8-9 are fulfilled.

Assume, that the trajectory is periodic with the period T . Denote the impact points by t_1, \dots, t_M , $0 < t_1 < \dots < t_M < T$. Consider the following function:

$$(31) \quad \Xi(\gamma) = \frac{\Psi(\gamma, t_1 - 0)}{\Psi(\gamma, 0)} \frac{\Psi(\gamma, t_2 - 0)}{\Psi(\gamma, t_1 + 0)} \dots \frac{\Psi(\gamma, T)}{\Psi(\gamma, t_M + 0)}$$

Lemma 3.7. *The function $\Xi(\gamma)$ has the following analytic properties:*

- (1) $\Xi(\gamma)$ is holomorphic in Γ outside the points $\lambda = 0$ and impact points D_s^\pm .
- (2)

$$(32) \quad \Xi(\gamma) = 1 + \frac{iT}{\sqrt{\lambda}} + \dots, \quad \text{as } \lambda \rightarrow \infty$$

- (3) There exists a constant L such, that

$$(33) \quad \exp\left(-\frac{iL}{\sqrt{\lambda}}\right) \Xi(\gamma)$$

is regular near $\lambda = 0$.

- (4) $\Xi(\gamma)$ has a pole of order \mathfrak{m}_s at the point D_s^{out} and a zero of order \mathfrak{m}_s at the point D_s^{in} where \mathfrak{m}_s denotes the number of impacts to the reflector $Q \cap Q_s$.

Let us introduce generalized quasimomentum \tilde{p} for the billiard motion. We would like to stress, that the generalized quasimomentum

does not coincide with the “true” quasimomentum for the non-smooth function $u(t)$ describing the billiard motion. The “true” quasimomentum is defined on an infinite-genus Riemann surface.

By definition:

$$(34) \quad \tilde{p} = \frac{1}{iT} \ln (\Xi(\gamma))$$

The function $\tilde{p}(\gamma)$ is multivalued in Γ but its differential $d\tilde{p}$ is well-defined.

Lemma 3.8. *The function $\tilde{p}(\gamma)$ is a single-valued function on $\Gamma^* = \Gamma \setminus \left(\bigcup_{j=1}^n \tilde{\alpha}_j \right)$. In other words Γ^* is obtained from Γ by removing the divisor trajectories.*

The proof of the Lemma follows immediately from the formula:

$$(35) \quad \tilde{p} = \frac{1}{iT} \left[\int_0^{t_1} \frac{\dot{\Psi}(\gamma, t)}{\Psi(\gamma, t)} dt + \int_{t_1}^{t_2} \frac{\dot{\Psi}(\gamma, t)}{\Psi(\gamma, t)} dt + \dots + \int_{t_M}^T \frac{\dot{\Psi}(\gamma, t)}{\Psi(\gamma, t)} dt \right]$$

The formula is well-defined if the denominator does not vanish.

Lemma 3.9. *Assume, that the gap $[b_{2j-1}, b_{2j}]$ contains at least one reflector, and that $\gamma \in \Gamma$ lies inside the gap but outside the divisor trajectory. Then $\operatorname{Re}(\Xi(\gamma)) = 0$.*

The proof follows immediately from the formula (35).

Now we are ready to formulate the properties of the generalized quasimomentum.

Lemma 3.10. (1) $d\tilde{p}$ is a meromorphic differential in Γ .

(2) $d\tilde{p}$ is holomorphic outside the points $\lambda = 0, D_s^{\text{in}}, D_s^{\text{out}}$.

(3) $d\tilde{p}$ has first-order poles at the points $D_s^{\text{in}}, D_s^{\text{out}}$ and

$$(36) \quad \operatorname{Res}_{D_s^{\text{in}}} d\tilde{p} = -\operatorname{Res}_{D_s^{\text{out}}} d\tilde{p}$$

If we have two reflectors d_s, d_{s+1} in the same gap, then

$$(37) \quad \operatorname{Res}_{D_s^{\text{in}}} d\tilde{p} = \operatorname{Res}_{D_{s+1}^{\text{in}}} d\tilde{p}.$$

(4) $d\tilde{p}$ has a second-order pole with zero residue at $\lambda = 0$.

(5)

$$(38) \quad d\tilde{p} = d \left(\frac{1}{\sqrt{\lambda}} \right) + O \left(\frac{1}{\lambda^2} \right) d\lambda, \quad \text{as } \lambda \rightarrow \infty.$$

(6)

$$(39) \quad \oint_{\alpha_j} d\tilde{p} = 0, \quad j = 1, \dots, n.$$

(7)

$$(40) \quad \oint_{\beta_j} d\tilde{p} = 0, \quad \text{if the gap } [b_{2j-1}, b_{2j}] \text{ contains at least one reflector.}$$

Let us recall that for gaps with reflectors the cycles β_j do not intersect the divisor trajectories.

(8)

$$(41) \quad \text{Res}_{D_s^{\text{in}}} d\tilde{p} = \frac{\mathfrak{m}_s}{iT}, \quad \mathfrak{m}_s \in \mathbb{N}$$

where \mathfrak{m}_s denotes the number of impacts to the obstacle $Q \cap Q_s$ for $0 \leq t < T$.

(9)

$$(42) \quad \oint_{\beta_j} d\tilde{p} = \frac{2\pi n_j}{T}, \quad n_j \in \mathbb{Z} \quad \text{if the gap } [b_{2j-1}, b_{2j}] \text{ contains no reflectors.}$$

Properties 6, 7 follow immediately from Lemmata 3.8, 3.9, and the fact that these cycles do not intersect the divisor trajectories is essential. All other properties are completely standard.

Remark 3.11. *Differential $d\tilde{p}$ is even with respect to the transposition of the sheets of Γ . Therefore the second term in (38) can be estimated more accurately:*

$$(43) \quad d\tilde{p} = d\left(\frac{1}{\sqrt{\lambda}}\right) + O\left(\frac{1}{\lambda^{5/2}}\right) d\lambda, \quad \text{as } \lambda \rightarrow \infty.$$

At $\lambda = 0$ one has the following expansion:

$$(44) \quad d\tilde{p} = \frac{L}{T} d\left(\frac{1}{\sqrt{\lambda}}\right) + O(1)d(\sqrt{\lambda}),$$

where L is the same as in (33).

Let us have a generic spectral curve Γ with a set of reflectors d_s (with the reality conditions, formulated above).

Lemma 3.12. *There exists an unique differential with the properties 1-7.*

This differential plays the role of the generalized quasimomentum differential for generic billiard spectral data, and we shall denote it with the same symbol $d\tilde{p}$.

Proof of Lemma 3.12. Let us construct $d\tilde{p}$ as the following linear combination:

$$(45) \quad d\tilde{p} = \mathcal{C}dp', \quad \text{where} \quad dp' = d\hat{p} + \sum_{k \in \mathcal{J}} \mathcal{D}_k \Omega_{(k)},$$

\mathcal{J} denotes the set of all α -cycles, containing reflectors, $d\hat{p}$ is the Abelian differential of the second kind with a second-order pole at the point $\lambda = 0$ such, that

$$(46) \quad d\hat{p} = d\left(\frac{1}{\sqrt{\lambda}}\right) + \dots \quad \text{as } \lambda \rightarrow 0, \quad \oint_{\alpha_j} d\hat{p} = 0, \quad j = 1, \dots, n,$$

$\Omega_{(k)}$ are the Abelian differentials of the third kind with first-order poles at the reflector points in the α_k ,

$$(47) \quad \text{Res}_{D_s^{(in)}} \Omega_{(k)} = +1, \quad \text{Res}_{D_s^{(out)}} \Omega_{(k)} = -1, \quad D_s^{(in)}, D_s^{(out)} \in \alpha_k,$$

$$(48) \quad \oint_{\alpha_j} \Omega_{(k)} = 0, \quad j = 1, \dots, n.$$

From the Riemann bilinear relations it follows, that

$$(49) \quad \mathbf{B}_{lk} = \oint_{\beta_l} \Omega_{(k)} = -2\pi i \int_{\tilde{\alpha}_k} \hat{\omega}_l.$$

By Corollary 3.5 the matrix \mathbf{B}_{lk} , $l, k \in \mathcal{J}$ is non-degenerate, therefore conditions (40) defines the constants \mathcal{D}_k uniquely.

Let us consider the divisor dynamics with reflections in Γ , written in terms of the KdV time σ (see (14)). Then outside the divisor trajectories we have:

$$(50) \quad dp' = \lim_{L \rightarrow \infty} \frac{1}{iL} \int_0^L \frac{\Psi'(\gamma, \sigma)}{\Psi(\gamma, \sigma)} d\sigma$$

For the logarithmic derivative of the wave function we have the following formula

$$(51) \quad \frac{\Psi'(\gamma, \sigma)}{\Psi(\gamma, \sigma)} = \frac{\sqrt{\prod_{j=1}^{2n} b_j}}{\prod_{j=1}^n \lambda_j(\sigma)} \cdot \frac{w + \lambda R_{n-1}(\lambda)}{\lambda(\lambda - \lambda_1(\sigma)) \dots (\lambda - \lambda_n(\sigma))},$$

where $R_{n-1}(\lambda)$ denotes some polynomial of degree $n - 1$. For $\lambda \rightarrow \infty$ we have

$$(52) \quad \frac{\Psi'(\gamma, \sigma)}{\Psi(\gamma, \sigma)} \sim \frac{\sqrt{\prod_{j=1}^{2n} b_j}}{\prod_{j=1}^n \lambda_j(\sigma)} \cdot \frac{i}{\sqrt{\lambda}},$$

therefore the leading term at ∞ does not vanish and has a constant sign. As a corollary the leading term of dp' at ∞ does not vanish, and the constant \mathcal{C} is defined by (38).

Lemma 3.12 is proved.

Let us formulate the sufficient conditions for the periodicity of the billiard trajectory.

Proposition 3.13. *Assume that the quasimomentum differential $d\tilde{p}$, defined by the properties 1-7 satisfy also properties 8, 9 for some T .*

Then the corresponding divisor billiard motion is periodic and with the period T in the natural parameter, and there exists an integer K such, that the corresponding billiard trajectory on the ellipsoid is periodic with the period KT .

Let us consider the divisor dynamics with reflections in Γ , written in terms of the KdV time σ (see (14)). Denote by $\tilde{A}(t)$ the Abel transform of the divisor expanded by the auxiliary basis $\tilde{\omega}_1, \dots, \tilde{\omega}_n$. From a standard calculation based on the Riemann bilinear identities it follows that

$$\frac{d\tilde{A}_j(\sigma)}{d\sigma} = \begin{cases} \frac{m_j}{L} & \text{if the gap } [b_{2j-1}, b_{2j}] \text{ contains reflector points} \\ \frac{n_j}{L} & \text{if the gap } [b_{2j-1}, b_{2j}] \text{ contains no reflectors.} \end{cases},$$

where L is the same as in (33), (44).

Let us denote by \mathcal{G}_1 the union of all trajectories generated by the divisor motion for $0 \leq \sigma \leq L$ and by \mathcal{G}_0 the following collection of oriented curves:

$$\mathcal{G}_0 = \left\{ \bigcup_{\text{all gaps containing reflectors}} m_j \tilde{\alpha}_j \right\} \cup \left\{ \bigcup_{\text{all gaps without reflectors}} n_j \tilde{\alpha}_j \right\}.$$

From the Dubrovin equations (5) it follows that the direction of motion in each gap is constant. Therefore the difference between \mathcal{G}_1 and \mathcal{G}_0 satisfies the conditions of Lemma 3.3. The Abel transform for this difference is zero, therefore these two sets of paths coincide. It implies the periodicity of the divisor motion with the period L in the variable σ .

Following the procedure described above one can naturally introduce an analog of the generalized quasimomentum with respect to the variable σ . Let us denote it by dp' . It coincides with $d\tilde{p}$ up to a normalization constant, and it is easy to check, that $dp' = \frac{T}{L}d\tilde{p}$, therefore the divisor motion is also periodic in t with the period T .

Remark 3.14. *Let us show, that both of the periodicity conditions formulated above are equivalent. Equation (29) defines a collection of numbers m_j uniquely up to a constant multiplier. From the bilinear relations we see that they are proportional to either residues or β -periods of the generalized quasimomentum differential $d\tilde{p}$ whether or not the corresponding gap contains reflectors. Proposition 3.13 means exactly that in the periodic case one can choose the constant multiplier so that all m_j are integer numbers.*

4. ISOPERIODIC DEFORMATIONS FOR THE BILLIARD MOTION

Denote the zeroes of the quasimomentum differential $d\tilde{p}$ by $\eta_1, \dots, \eta_{n+r}$. Then the quasimomentum differential takes the form:

$$(53) \quad d\tilde{p} = -\frac{1}{2} \frac{\prod_{j=1}^{n+r} (\lambda - \eta_j)}{\lambda \prod_{s=1}^r (\lambda - d_s) \sqrt{\lambda \prod_{j=1}^{2n} (\lambda - b_j)}} d\lambda$$

By analogy with [22] we shall use the following statement:

Lemma 4.1. *Assume, that we have a periodic spectral data for the billiard motions and a deformation of these data such, that*

$$(54) \quad \left. \frac{\partial d\tilde{p}}{\partial \zeta} \right|_{\lambda=\text{const}} = dP(\gamma),$$

or, equivalently,

$$(55) \quad \left. \frac{\partial \tilde{p}}{\partial \zeta} \right|_{\lambda=\text{const}} = P(\gamma),$$

where $P(\gamma)$ is a meromorphic function in Γ . Then this deformation preserves the T -periodicity of the divisor motion associated to the billiard dynamics.

In our situation the spectral curve depends on the variable ζ . While calculating the partial derivatives with respect to ζ we assume the projection of the point γ to the λ -plane to remain constant. We remark that this rule generates singularities at the moving branch points.

Remark 4.2. *The function \tilde{p} in the left-hand side of (55) is multivalued. The condition that its derivative is single-valued is equivalent to the isoperiodicity condition (see [22]).*

The proof of the Lemma is rather straightforward. From the equation (54) it follows that all periods of $d\tilde{p}$ as well as the residues at the points $D_s^{\text{in}}, D_s^{\text{out}}$ are constant, therefore the properties 8, 9 are satisfied identically.

Lemma 4.3. *Assume, that we have a deformation of the periodic spectral data satisfying (54). Then $P(\gamma)$ has the following analytic properties:*

- (1) $P(\gamma)$ has 1-st order poles at the points $\lambda = 0$ and $D_s^{\text{in}}, D_s^{\text{out}}$.
- (2) $P(\gamma)$ has 1-st order poles at the finite ramification points b_1, \dots, b_{2n} .
- (3) $P(\gamma)$ has a third-order zero at $\lambda = \infty$. (Here we use that $d\tilde{p}$ is even with respect to the hyperelliptic involution τ).
- (4) $P(\gamma)$ is odd with respect to the hyperelliptic involution τ : $P(\tau\gamma) = -P(\gamma)$.

The proof again is rather straightforward. We prove now that the opposite is also true.

Lemma 4.4. *Assume that $P(\gamma)$ is an arbitrary meromorphic function satisfying the properties 1-3 from Lemma 4.3. Then it generates an infinitesimal isoperiodic deformation.*

This lemma generalizes the analogous statement from [22]. The idea how to construct this deformation is very similar to the arguments of the paper [20].

First of all let us define the dynamics of the branch points. Consider the expansions of \tilde{p} , P near the point b_j :

$$(56) \quad \tilde{p} = \tilde{p}_{0,b_j}(\zeta) + \tilde{p}_{1,b_j}(\zeta)\sqrt{\lambda - b_j(\zeta)} + \dots \quad d\tilde{p} = \left[\frac{\tilde{p}_{1,b_j}(\zeta)}{2\sqrt{\lambda - b_j(\zeta)}} + O(1) \right] d\lambda$$

$$(57) \quad P = \frac{P_{-1,b_j}(\zeta)}{\sqrt{\lambda - b_j(\zeta)}} + \dots$$

$$(58) \quad \left. \frac{\partial \tilde{p}}{\partial \zeta} \right|_{\lambda=\text{const}} = -\frac{1}{2\sqrt{\lambda - b_j(\zeta)}} \tilde{p}_{1,b_j}(\zeta) \frac{\partial b_j(\zeta)}{\partial \zeta} + \dots$$

Comparing (57) with (58) we obtain:

$$(59) \quad \frac{\partial b_j(\zeta)}{\partial \zeta} = -2 \frac{P_{0,d_s}(\zeta)}{\tilde{p}_{1,b_j}(\zeta)}$$

Let us define now the dynamics of the reflectors d_s . Near the points $D_s^{\text{in}}, D_s^{\text{out}}$ we have respectively:

$$(60) \quad \tilde{p} = \pm \frac{\mathfrak{m}_s}{iT} \ln(\lambda - d_s(\zeta)) + \dots \quad d\tilde{p} = \pm \frac{\mathfrak{m}_s}{iT(\lambda - d_s(\zeta))} d\lambda + \dots$$

$$(61) \quad P = \pm \frac{P_{-1,d_s}(\zeta)}{(\lambda - d_s(\zeta))} + \dots$$

$$(62) \quad \left. \frac{\partial \tilde{p}}{\partial \zeta} \right|_{\lambda=\text{const}} = \mp \frac{\mathfrak{m}_s}{iT(\lambda - d_s(\zeta))} \frac{\partial d_s(\zeta)}{\partial \zeta} + \dots$$

Comparing (61) with (62) we obtain:

$$(63) \quad \frac{\partial d_s(\zeta)}{\partial \zeta} = - \frac{iTP_{-1,d_s}(\zeta)}{\mathfrak{m}_s}$$

Consider the deformation defined by (59), (63). Then equations (54), (55) define a deformation of $\tilde{p}(\gamma)$ as well as a deformation of $d\tilde{p}$ such, that the deformed function $d\tilde{p}$ is defined on the new spectral curve and satisfies all properties of Lemma 3.10. Therefore the deformed spectral data also generate periodic divisor motions with the same number of impacts.

By analogy with [22] it is also convenient to derive the dynamics for the zeroes of $d\tilde{p}$. In the neighborhood of η_j we have

$$(64) \quad d\tilde{p} = \pm \tilde{p}_{1,\eta_j}(\zeta)(\lambda - \eta_j(\zeta))d\lambda + \dots$$

$$(65) \quad P = \pm [P_{0,\eta_j}(\zeta) + P_{1,\eta_j}(\zeta)(\lambda - \eta_j(\zeta))] + \dots, \quad dP = \pm P_{1,\eta_j}(\zeta)d\lambda + \dots$$

$$(66) \quad \left. \frac{\partial d\tilde{p}}{\partial \zeta} \right|_{\lambda=\text{const}} = \mp \left[\tilde{p}_{1,\eta_j}(\zeta) \frac{\partial \eta_j(\zeta)}{\partial \zeta} \right] d\lambda + \dots$$

$$(67) \quad \frac{\partial \eta_j(\zeta)}{\partial \zeta} = - \frac{P_{1,\eta_j}(\zeta)}{\tilde{p}_{1,\eta_j}(\zeta)}$$

We have shown that we have a one-to-one correspondence between the period preserving deformations and meromorphic functions with special analytic properties. By analogy with [22] let us introduce a convenient basis in the space of such meromorphic functions. Let us define:

$$(68) \quad P_j(\gamma) = -\mathcal{C}_j \frac{\lambda d\tilde{p}(\gamma)}{(\lambda - \eta_j)d\lambda}, \quad j = 1, \dots, n+r.$$

In the paper by Marchenko and Ostrovski [31] the following coordinates were used on the space of spectral curves associated with periodic potentials:

$$(69) \quad \zeta_j = -ip(\eta_j), \quad j = 1, \dots, g,$$

where η_j are the zeroes of the quasimomentum differential. It is convenient to use these coordinates in the method of isoperiodic deformation. In particular, the corresponding flows commute at regular points for these flows (all zeroes η_j are pairwise distinct).

This choice of local coordinates correspond to the following choice of the factors \mathcal{C}_j :

$$(70) \quad P_j(\eta_j) = i.$$

We use here, that

$$(71) \quad \left. \frac{\partial \tilde{p}}{\partial \zeta} \right|_{\lambda=\text{const}, \gamma=\eta_j} = \frac{\partial \tilde{p}(\eta_j)}{\partial \zeta}$$

at the zeroes of the generalized quasimomentum differential $d\tilde{p}$. Therefore we obtain

$$(72) \quad \mathcal{C}_j^2 = 2 \frac{-\eta_j \prod_{k=1}^{2n} (\eta_j - b_k) \prod_{s=1}^r (\eta_j - d_s)^2}{\prod_{k \neq j} (\eta_j - \eta_k)^2}$$

It is convenient to assume $\mathcal{C}_j > 0$ for real zeros of quasimomentum differential.

Calculating the leading terms of $P_j(\gamma)$ at the branch points, reflectors and zeroes different from η_j we obtain:

$$(73) \quad \begin{aligned} \frac{\partial b_k}{\partial \zeta_j} &= \mathcal{C}_j \frac{b_k}{b_k - \eta_j} \\ \frac{\partial d_s}{\partial \zeta_j} &= \mathcal{C}_j \frac{d_s}{d_s - \eta_j} \\ \frac{\partial \eta_k}{\partial \zeta_j} &= \mathcal{C}_j \frac{\eta_k}{\eta_k - \eta_j}, \quad k \neq j \\ \frac{\partial \eta_j}{\partial \zeta_j} &= \mathcal{C}_j \eta_j \left[\sum_{k \neq j} \frac{1}{\eta_j - \eta_k} - \frac{1}{2\eta_j} - \frac{1}{2} \sum_{k=1}^{2n} \frac{1}{\eta_j - b_k} - \sum_{s=1}^r \frac{1}{\eta_j - d_s} \right]. \end{aligned}$$

Summarizing all these results we obtain:

Theorem 4.5. *Let us consider a generic set of spectral data corresponding to a periodic billiard motion (generic means that all zeroes of $d\tilde{p}$ are pairwise distinct). Then equations (73) with the normalization*

(72) locally define a set of $n + r$ commuting flows on the spectral data, preserving the T -periodicity of the billiard motion as well as the number of impacts during a period.

Remark 4.6. *In the billiard motion it is critical to respect the reality conditions. To keep the reality of the spectral data it is sufficient to assume, that the times ζ_j corresponding to real zeroes η_j of $d\tilde{p}$ are real, the times corresponding to complex conjugate pairs of zeroes η_k , $\eta_l = \overline{\eta_k}$ are complex conjugate: $\zeta_l = \overline{\zeta_k}$.*

5. APPENDIX. SOME DETAILS OF NUMERIC CHECK.

To illustrate the approach developed above, some numerical simulation were made. For computational purposes it is more convenient to use the spectral parameter $E = 1/\lambda$ instead of λ . Equations (73) were implemented using the standard 5-th order Runge-Kutta method. The authors used the integrator, developed by Hairer, Nørsett and Wanner [24], this free software package is available on the Ernst Hairer's web site <http://www.unige.ch/~hairer/>. The program was developed as a modification of the program used in [22] to check isoperiodic approach to the KdV equation.

The novelty of the problem was, that the calculation of the starting point for the billiard problem required some additional calculations. Let us discuss the search of the starting point.

It is convenient to calculate the generalized quasimomentum differential for a small periodic perturbation of the constant potential $u(x) = 1$. We assume the period $T = 1$. In our experiments we assumed that we have one reflector in the last gap $[b_{2n-1}, b_{2n}]$. In our program the branch points are enumerate in the inverse order $E_1 = 1/b_{2n}, \dots, E_{2n} = 1/b_1$. For numerical reasons it is critical to make the gap, containing the reflector, a little bigger than the other ones. We also assume, that at the starting point the reflection wall lies in the center of the distinguished gap. The starting lengths are: $E_2 - E_1 \sim 10^{-6}$ and $E_{2k} - E_{2k-1} \sim 10^{-9}$, $k > 1$.

First of all, let us calculate the positions of the point r (the center of the gap), and the zeroes $A_0 = 1/\eta_0$, $A_1 = 1/\eta_1$ (up to some higher order corrections).

Let

(74)

$$E_1 = C - A, \quad E_2 = C + A, \quad 1/d = R = C, \quad A_0 = C + \tilde{A}_0, \quad A_1 = C + \tilde{A}_1, \quad A \ll 1.$$

Assuming all other gaps to have zero length (genus 1 approximation) we have

$$d\tilde{\omega}_1 = \varkappa \frac{1}{\sqrt{E(E-E_1)(E-E_2)}} dE$$

$$\mathbf{C} \sim \frac{\pi^2}{4}$$

Let us denote

$$E = Z + \mathbf{C}$$

The constant \varkappa is defined from the condition

$$\int_{Z=0}^{Z=\mathbf{A}} \tilde{\omega}_1 = \frac{1}{2}.$$

Expanding $d\tilde{\omega}_1$ near $Z = 0$ we obtain:

$$d\tilde{\omega}_1 = \varkappa \frac{1}{\sqrt{\mathbf{C} + Z}} \frac{dZ}{\sqrt{Z^2 - \mathbf{A}^2}} = \frac{\varkappa}{\sqrt{\mathbf{C}}} \left(1 - \frac{Z}{2\mathbf{C}} + O(Z^2) \right) \frac{dZ}{\sqrt{Z^2 - \mathbf{A}^2}}$$

Using the table integrals

$$\int_{Z=0}^{Z=\mathbf{A}} \frac{dZ}{\sqrt{Z^2 - \mathbf{A}^2}} = \frac{\pi i}{2}$$

$$\int_{Z=0}^{Z=\mathbf{A}} \frac{Z dZ}{\sqrt{Z^2 - \mathbf{A}^2}} = i\mathbf{A}$$

we obtain:

$$\int_{Z=0}^{Z=\mathbf{A}} \tilde{\omega}_1 = \frac{\varkappa}{\sqrt{\mathbf{C}}} \times \left(\frac{\pi i}{2} - \frac{i\mathbf{A}}{2\mathbf{C}} + O(\mathbf{A}^2) \right).$$

Therefore

$$\varkappa = \frac{\sqrt{\mathbf{C}}}{i \left(\pi - \frac{\mathbf{A}}{\mathbf{C}} \right)} + O(\mathbf{A}^2)$$

In the genus 1 approximation we have

$$d\tilde{p} = \frac{\mathbf{B}}{2} \frac{(E - A_0)(E - A_1)}{(E - \mathbf{R})\sqrt{E(E - E_1)(E - E_2)}} dE.$$

Let us look at the conditions on \tilde{A}_0, \tilde{A}_1 imposed by

$$\oint_{\alpha_n} d\tilde{p} = 0$$

Expanding near $Z = 0$ we obtain

$$\begin{aligned} d\tilde{p} &= \frac{B}{2\sqrt{C}} \left(1 - \frac{Z}{2C} + O(Z^2) \right) \left(\frac{(Z - \tilde{A}_0)(Z - \tilde{A}_1)}{Z} \right) \frac{dZ}{\sqrt{Z^2 - A^2}} = \\ &= \frac{B}{2\sqrt{C}} \left(\frac{\tilde{A}_0\tilde{A}_1}{Z} - \left(\tilde{A}_0 + \tilde{A}_1 + \frac{\tilde{A}_0\tilde{A}_1}{2C} \right) + O(Z) \right) \frac{dZ}{\sqrt{Z^2 - A^2}} \end{aligned}$$

Due to parity constraints

$$\oint_{\alpha_n} \frac{1}{Z} \frac{dZ}{\sqrt{Z^2 - A^2}} = \oint_{\alpha_n} Z \frac{dZ}{\sqrt{Z^2 - A^2}} = 0$$

and finally we obtain:

$$(75) \quad \tilde{A}_0 + \tilde{A}_1 + \frac{\tilde{A}_0\tilde{A}_1}{2C} = O(A^2)$$

It is difficult to use the condition

$$\oint_{\beta_n} d\tilde{p} = 0$$

directly. It is more convenient to use Riemann bilinear relations instead. We know, that if

$$\oint_{\beta_n} d\tilde{p} = \oint_{\alpha_n} d\tilde{p} = 0,$$

then

$$\text{Res}_{R^{\text{in}}} d\tilde{p} = \text{Res}_{\infty} \tilde{p}\tilde{\omega}_1$$

Due to the periodicity condition

$$\text{Res}_{R^{\text{in}}} d\tilde{p} = -i,$$

i.e.

$$\text{Res}_{\infty} \tilde{p}\tilde{\omega}_1 = -i$$

But

$$\tilde{p}\tilde{\omega}_1 = \frac{dE}{E} \left(1 + O\left(\frac{1}{E}\right) \right) B\mathcal{K}$$

therefore

$$(76) \quad B\mathcal{K} = \frac{i}{2}$$

$$B = \frac{(-C)\sqrt{C^2 - A^2}}{C^2 + C(\tilde{A}_0 + \tilde{A}_1) + \tilde{A}_0\tilde{A}_1} = \frac{-C^2}{C^2 + \frac{\tilde{A}_0\tilde{A}_1}{2}} + O(A^2) = -\frac{1}{1 + \frac{\tilde{A}_0\tilde{A}_1}{2C^2}} + O(A^2)$$

Direct calculation of the residues of $d\tilde{p}$ at the points $R^{\text{in}}, R^{\text{out}}$, gives us

$$1 = B \frac{\tilde{A}_0 \tilde{A}_1}{2\sqrt{C}A}$$

Therefore

$$(77) \quad \tilde{A}_0 \tilde{A}_1 \sim -2A\sqrt{C} \left(1 + \frac{\tilde{A}_0 \tilde{A}_1}{2C^2} \right) \sim -2A\sqrt{C} \left(1 - \frac{A}{C\sqrt{C}} \right)$$

$$(78) \quad \tilde{A}_0 + \tilde{A}_1 \sim \frac{A}{\sqrt{C}} \left(1 - \frac{A}{C\sqrt{C}} \right)$$

To determine C let us use (76).

$$\begin{aligned} \frac{\sqrt{C}}{\left(\pi - \frac{A}{C}\right)} \times \frac{1}{1 + \frac{\tilde{A}_0 \tilde{A}_1}{2C^2}} &= \frac{1}{2} \\ \frac{\sqrt{C}}{\left(\pi - \frac{A}{C}\right)} \times \frac{1}{1 - \frac{A}{C\sqrt{C}}} &= \frac{1}{2} \end{aligned}$$

Finally, we obtain

$$(79) \quad C = \left[\left(\frac{\pi - \frac{A}{C}}{2} \right) \left(1 - \frac{A}{C\sqrt{C}} \right) \right]^2$$

To find the starting point we first solve (79), then we calculate \tilde{A}_0, \tilde{A}_1 using (77), (78)

Let us calculate the resonant points, defined by the condition

$$\tilde{p}(C_k) = \pi k, \quad k > 1.$$

To do it let us expand $d\tilde{p}$ outside the neighborhood of the point C .

$$\begin{aligned} d\tilde{p} &= \frac{B}{2} \frac{dZ}{\sqrt{Z+C}\sqrt{Z^2-A^2}} \frac{(Z-\tilde{A}_0)(Z-\tilde{A}_1)}{Z} = \\ &= \frac{B}{2} \frac{dZ}{\sqrt{Z+C}} \frac{1}{\sqrt{Z^2-A^2}} \left(\frac{\tilde{A}_0 \tilde{A}_1}{Z} - (\tilde{A}_0 + \tilde{A}_1) + Z \right) = \\ &= \frac{B}{2} \frac{dZ}{\sqrt{Z+C}} \left(\frac{1}{Z} + O(A^2) \right) \left(\frac{\tilde{A}_0 \tilde{A}_1}{Z} - (\tilde{A}_0 + \tilde{A}_1) + Z \right) = \\ &= \frac{B}{2} \frac{dZ}{\sqrt{Z+C}} \left(1 - \frac{\tilde{A}_0 + \tilde{A}_1}{Z} + \frac{\tilde{A}_0 \tilde{A}_1}{Z^2} + O(A^2) \right) = \\ &= \frac{B}{2} \frac{dE}{\sqrt{E}} \left(1 - \frac{\tilde{A}_0 + \tilde{A}_1}{E-C} + \frac{\tilde{A}_0 \tilde{A}_1}{(E-C)^2} + O(A^2) \right) \end{aligned}$$

Using the table integrals:

$$\begin{aligned}\int \frac{dE}{2\sqrt{E}} &= \sqrt{E} \\ \int \frac{dE}{2\sqrt{E}(E - \mathfrak{C})} &= \frac{1}{2\sqrt{\mathfrak{C}}} \log \left(\frac{\sqrt{E} - \sqrt{\mathfrak{C}}}{\sqrt{E} + \sqrt{\mathfrak{C}}} \right) \\ \int \frac{dE}{2\sqrt{E}(E - \mathfrak{C})^2} &= -\frac{1}{4\mathfrak{C}} \left[\frac{2\sqrt{E}}{E - \mathfrak{C}} + \frac{1}{\sqrt{\mathfrak{C}}} \log \left(\frac{\sqrt{E} - \sqrt{\mathfrak{C}}}{\sqrt{E} + \sqrt{\mathfrak{C}}} \right) \right]\end{aligned}$$

we get

$$\tilde{p} = \mathbf{B} \left[\sqrt{E} - \left(\tilde{A}_0 + \tilde{A}_1 + \frac{\tilde{A}_0 \tilde{A}_1}{2\mathfrak{C}} \right) \frac{1}{2\sqrt{\mathfrak{C}}} \log \left(\frac{\sqrt{E} - \sqrt{\mathfrak{C}}}{\sqrt{E} + \sqrt{\mathfrak{C}}} \right) - \frac{\tilde{A}_0 \tilde{A}_1}{2\mathfrak{C}} \frac{\sqrt{E}}{E - \mathfrak{C}} + O(\mathbf{A}^2) \right]$$

Taking into account (75) we obtain

$$\tilde{p} = \mathbf{B} \sqrt{E} \left[1 - \frac{\tilde{A}_0 \tilde{A}_1}{2\mathfrak{C}} \frac{1}{E - \mathfrak{C}} + O(\mathbf{A}^2) \right]$$

Therefore we calculate the resonant points from the equation

$$(80) \quad \mathfrak{C}_k = \left[\frac{\pi k}{\mathbf{B}} \frac{1}{1 - \frac{\tilde{A}_0 \tilde{A}_1}{2\mathfrak{C}(\mathfrak{C}_k - \mathfrak{C})}} \right]^2$$

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